Chaos around the superposition of a monopole and a thick disk

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Abstract

We extend recent investigations on the integrability of oblique orbits of test particles under the gravitational field corresponding to the superposition of an infinitesimally thin disk and a monopole to the more realistic case, for astrophysical purposes, of a thick disk. Exhaustive numerical analyses were performed and the robustness of the recent results is confirmed. We also found that, for smooth distributions of matter, the disk thickness can attenuate the chaotic behavior of the bounded oblique orbits. Perturbations leading to the breakdown of the reflection symmetry about the equatorial plane, nevertheless, may enhance significantly the chaotic behavior, in agreement with recent studies on oblate models.

95.10.Fh, 05.45.-a

Typeset using REVTEX

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The recent observational evidences suggesting that huge black-holes might inhabit the center of many active galaxies [1] have motivated some investigations on the dynamics of test particles in gravitational systems consisting in the superposition of monopoles and disks. Infinitesimally thin disks are frequently used to model flattened galaxies [2]. Some exact relativistic solutions describing the superposition of non-rotating black-holes and static thin disk, and their respective Newtonian limits, were presented and discussed in [3]. In [4], the integrability of oblique orbits of test particles around the exact superposition of a black-hole and an infinitesimally static thin disk was considered. Bounded zones of chaotic behavior were found for both the relativistic and Newtonian limits. There are several examples in the literature of chaotic motion involving black-holes: in the fixed two centers problem [5–7], in a black-hole surrounded by gravitational waves [8,9], and in several core–shell models with relevance to the description of galaxies (see [10] for a recent review). As to the Newtonian case, we notice, for instance, the recent work of C. Chicone, B. Mashhoon, and D. G. Retzloff on the chaotic behavior of the Hill system [11].

The Newtonian analysis of [4] has revealed an interesting property of the dynamics of oblique bounded orbits around the superposition of a monopole and an infinitesimally thin disk. Since one was manly interested in bounded motions close to the monopole, it was assumed that the disk was infinite and homogeneous. This situation corresponds the simplest superposition of a monopole and a disk. Using cylindrical coordinates (r, θ, z) with the monopole, with mass M, located in the origin and the disk corresponding to the plane $z = 0$, the gravitational potential is given by

$$V(r, \theta, z) = -\frac{M}{\sqrt{r^2 + z^2}} + \alpha|z|, \tag{1}$$

where α is a positive parameter standing for the superficial mass density of the disk. The angular momentum L in the z direction is conserved, and we can easily reduce the three-dimensional original problem to a two-dimensional one in the coordinates (r, z) with the Hamiltonian given by

$$H = \frac{\dot{r}^2}{2} + \frac{\dot{z}^2}{2} + \frac{L^2}{2r^2} - \frac{M}{\sqrt{r^2 + z^2}} + \alpha|z|. \tag{2}$$

The Hamiltonian (2) is smooth everywhere except on the plane $z = 0$. Moreover, the parts of the trajectories restricted to the regions $z > 0$ and $z < 0$ are integrable [4]. The corresponding Hamilton-Jacobi equations restricted to these regions can be properly separated in parabolic coordinates, leading, respectively, to the second constants of motion

$$C_{z>0} = R_z - \alpha\frac{r^2}{2} \tag{3}$$

and

$$C_{z<0} = R_z + \alpha\frac{r^2}{2}, \tag{4}$$

where R_z is the z component of the Laplace-Runge-Lenz vector. Note that C is not smoothly defined on the disk. With the two constant of motions H and C, the equations for the trajectories of test particles, restricted to the regions $z > 0$ and $z < 0$, can be properly reduced to quadratures in parabolic coordinates [12,13]. A complete bounded oblique trajectory, therefore, corresponds to the matching of an infinite number of integrable trajectory pieces. Hence, the widespread zones of chaotic motion detected in [4] have their origin in the changes in the value of the constant C when the test particle crosses the disk $z = 0$. As to the relativistic case, in contrast, the trajectory pieces that do not cross the disk are themselves non-integrable, leading to typically larger chaotic regions than in the corresponding Newtonian limit [4].

Here, we study the robustness of the results for the Newtonian case by considering the more realistic case, for the description of flattened galaxies, of a superposition of a central monopole and a smooth thick disk with the potential (1) as the vanishing disk thickness limit. A smooth distribution of matter is considered as a disk if its radial gradients are much smaller than its vertical ones. A minimally realistic model for a rotating thick disk should obey Emden's equation for the stability of rotating polytropes [2]. As in [4], we will neglect the radial gradients, and, in this case, Emden's equation for the disk matter density $\rho(z)$ states that ($G = 1$)

$$\kappa\lambda\rho^{\lambda-2}\rho'' + \kappa\lambda(\lambda - 2)\rho^{\lambda-3}(\rho')^2 = -4\pi\rho, \tag{5}$$

where κ is the parameter relating the pressure to the matter density in the polytropic equation of state and $\lambda = 1 + 1/n$, n being the polytrope index. The matter density ρ is assumed to obey Poisson's equation $\nabla^2 V_{\rm D} = 4\pi\rho$. For the isothermal case ($\lambda = 1$), equation (5) admits as a solution the following distribution of matter

$$\rho(z) = \frac{\alpha}{4\pi z_0}\mathrm{sech}^2\frac{z}{z_0}, \tag{6}$$

which corresponds to the potential

$$V_{\rm D}(z) = \alpha z_0 \ln\cosh\frac{z}{z_0}, \tag{7}$$

where z_0 measures the disk "thickness", and α its "superficial" mass density. They obey the relation $2\kappa = \alpha z_0$. Typically, for realistic models of a rotating dust disk one has $z_0^2 \propto \langle V_z^2\rangle$ [14]. The matter density (6) corresponds, therefore, to a stable and smooth distribution of rotating matter concentrated on the plane $z = 0$. Moreover, in the limit of $z_0 \to 0$, we recover from (7) the infinitesimal thin disk potential $V_{\rm D}(z) = \alpha|z|$ and the corresponding δ distribution of matter from (6).

Thus, the dynamics of test particle moving around the superposition of our smooth thick disk and a monopole will be governed by the following smooth Hamiltonian

$$H = \frac{\dot r^2}{2} + \frac{\dot z^2}{2} + \frac{L^2}{2r^2} - \frac{M}{\sqrt{r^2+z^2}} + \alpha z_0 \ln\cosh\frac{z}{z_0}. \tag{8}$$

We wish to stress that the potential in (8) corresponds, indeed, to a first approximation of a realistic superposition of a monopole and a smooth thick disk with matter distribution given by (6). The whole superposition must also obey Emden's equation; in the present case it fails to obey at the origin. We are neglecting the radial gradients of the matter distributions caused by the stresses induced by the central monopole.

The Hamiltonian (8) does not belong to the class of integrable two-dimensional potentials with a second constant of motion polynomial in the momenta [15]. We will present strong evidences that (8) does not have a second constant of motion at all. We could solve numerically the system governed by (8) with great accuracy. Figure 1 shows a typical Poincaré's section ($H = -0.2, L = M = 1, \alpha = 0.1, z_0 = 1.5$) across the plane $z = 0$ revealing a widespread chaotic behavior. The disk thickness z_0, in this case, has the same magnitude of the typical z-amplitude of the trajectories. Figure 2 shows a sequence of low-energy sections ($H = -0.3, L = M = 1, \alpha = 0.1, z_0 = 0.0$ (a), 0.1 (b), 0.25 (c), and 0.5 (d)), constructed from the same trajectory initial conditions, where the attenuation of the chaotic behavior due to the disk thickness can be clearly appreciated. We could obtain thousands of intersections for each trajectory with a cumulative error, measured by the constant H, inferior to 10^{-12}. We notice that the equation of motion are invariant under the following rescalings:

$$
\begin{aligned}
r \to \lambda r, \qquad & z \to \lambda z, \qquad & t \to \lambda^{3/2} t, \\
\alpha \to \lambda^{-2}\alpha, \qquad & M \to M, \qquad & z_0 \to \lambda z_0 \\
H \to \lambda^{-1} H, \qquad & L \to \lambda^{1/2} L; &
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
r \to \lambda' r, \qquad & z \to \lambda' z, \qquad & t \to \lambda' t, \\
\alpha \to \lambda'^{-1}\alpha, \qquad & M \to \lambda' M, \qquad & z_0 \to \lambda' z_0 \\
H \to H, \qquad & L \to \lambda' L; &
\end{aligned} \tag{10}
$$

$\lambda > 0$ and $\lambda' > 0$, leading that, for each triple of nonzero parameters (M, α, z_0), one has, in fact, only one free parameter, namely $z_0\sqrt{\alpha/M}$.

The limit of z_0 much larger than the typical z-amplitude of the trajectories deserves a special attention. For this case, the Hamiltonian can be well approximated by

$$
H_\infty = \frac{\dot{r}^2}{2} + \frac{\dot{z}^2}{2} + \frac{L^2}{2r^2} - \frac{M}{\sqrt{r^2+z^2}} + \frac{\beta}{2} z^2, \tag{11}
$$

where $\beta = \alpha/z_0$. Such a potential is related to the potential of the Kepler problem perturbed by a quadrupole halo potential considered in [10]. Figure 3 shows a typical Poincaré's section ($H_\infty = -0.15, L = M = 1, \beta = 0.1$) across the plane $z = 0$ revealing a widespread chaotic behavior for the system governed by (11). We could also obtain thousands of intersections for each trajectory with a cumulative error inferior to 10^{-12}. Like in the infinitesimally thin disk case [4], due to the existence of two rescaling invariances, this Poincaré's section can be obtained for any non-zero values of β and M.

Our numerical finds confirm the robustness of the results presented in [4]. The chaotic behavior of bounded orbits can be considered as inherent to any system consisting in the superposition of a disk and a central monopole. We stress that the superpositions we have considered are symmetric under the reflection about the equatorial plane. A perturbation leading to the breakdown of the reflection symmetry could increase significantly the chaotic behavior, as the recent studies on oblate models have suggested [10]. We could indeed check such fact by considering a dipole-like perturbation of the Hamiltonian (8)

$$\tilde{H} = H - D\frac{z}{(r^2+z^2)^{3/2}}. \tag{12}$$

Figure 4 presents a typical Poincaré's surface ($\tilde{H} = -0.3, L = M = 1, \alpha = 0.1, z_0 = 0.25, D = 5\times 10^{-2}$) for this case. The zones of chaotic motion are larger than in the corresponding case for which $D = 0$ (Fig 2.c). However, the central family of regular orbits, if it does exit for $D = 0$, seems to be robust against dipole-like perturbations. Analogous conclusions hold also for the large z_0 case.

ACKNOWLEDGMENTS

The author is grateful to CNPq and FAPESP for the financial support, and to Prof. P.S. Letelier and R. Venegeroles for stimulating discussions.

FIG. 1. Poincaré's section $(r, \dot{r})$ across the plane $z = 0$ for oblique orbits, with $H = -0.2$ and $L = 1$, around the superposition of a monopole with mass $M = 1$ and a smooth disk with surface mass density $\alpha = 0.1$ and thickness $z_0 = 1.5$.

FIG. 2. Sequence of Poincaré's section $(r, \dot{r})$ across the plane $z = 0$ for oblique orbits, with $H = -0.3$ and $L = 1$, around the superposition of a monopole with mass $M = 1$ and a smooth disk with mass surface density $\alpha = 0.1$ and thickness $z_0 = 0.0$ (a), $z_0 = 0.1$ (b), $z_0 = 0.25$ (c), and $z_0 = 0.5$ (d). The sections were constructed from the same trajectory initial conditions.

FIG. 3. Poincaré's section $(r, \dot{r})$ across the plane $z = 0$ for oblique orbits, with $H_\infty = -0.15$ and $L = 1$, around the superposition of a monopole with mass $M = 1$ and a smooth disk of large thickness with $\beta = 0.1$.

FIG. 4. Poincaré's section $(r, \dot{r})$ across the plane $z = 0$ for oblique orbits, with $\tilde{H} = -0.3$ and $L = 1$, around the superposition of a central source characterized by a monopole charge $M = 1$ and a dipole strength $D = 5 \times 10^{-2}$, and a smooth disk with mass surface density $\alpha = 0.1$ and thickness $z_0 = 0.25$. The case corresponding to $D = 0$ is presented in Fig. 2.c.

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